Construction of 3D wormhole supported by phantom energy

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Abstract

In this article, we have found a series solution of 3D Einstein equations describing a wormhole for an inhomogeneous distribution of phantom energy. Here, we assume equation of state is linear but highly anistropic.

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Pure gravity in (2+1) dimensions is fascinating in its own right. In this case gravity does not propagate outside the sources i.e. no gravity out side matter. That means matter curves spacetime only locally. In other words, there is no gravitational waves. In (2+1) dimensional spacetime, Newtonian theory can not be obtained as a limit of Einstein's theory. Wormhole structure can not be obtained from Newtonian gravity. But it is wel-known that Einstein's general theory of relativity admits wormhole structure in spacetime since according to Einstein's theory, the presence of matter twists the geometric fabric of spacetime. For this reason, why we consider (2+1) dimensions to discuss wormhole structure. Now, wormholes are classical or quantum solutions for the gravitational field equations describing a bridge between two asymptotic manifolds. Classically, they can be interpreted as instantons describing a tunneling between two distant regions. In a pioneer work, Morris and Throne [1] have shown that the construction of wormhole would require a very unusual form of stress energy tensor. The matter that characterized the above stress energy tensor is known as exotic matter. This exotic i.e. hypothetical matter can be of the following form either energy density of matter $\rho < 0$ or $\rho > 0$ but pressure $p < 0$. In 21st century, this concept of negative energy is not pure fantasy, some of its effects have been produced in the laboratory [2]. Recent astronomical and cosmological observations indicate that the Universe is undergoing a phase of accelerated expansion [3]. Theoretical Physicists beleive that this acceleration is due to some unusual source of matter with positive energy density $\rho > 0$ and with negative pressure $p < 0$. This unsual matter source with the property, energy density, $\rho > 0$ but pressure $p < 0$ is known as phantom energy . Some how, if an advanced engineer could able to collect this unusual source of matter i.e phantom energy, then it would be possible to construct a wormhole. If wormhole could be constructed , the faster than light travel would be possible in other words, the time machine might be constructed. Now scientists are interested to know how much negative energy is needed to construct a wormhole. In 2003, Visser , Kar and Dadhich [4] have proposed that wormholes could be constructed with arbitrary small quantities of exotic matter. In recent past, Delgaty et al [5] have studied traversable wormhole in (2+1) dimension with a cosmological constant. In this article, we are decided to provide a prescription of 3D wormhole geometry by using phantom energy as source. One could imagine that an advanced engineer may use these results to construct and sustain a traversable wormhole so that the interstellar distances to be travelled in very short times. And we think, this might go forward our world.

As our target to provide a mathematical prescription of wormhole geometry in (2+1) dimensions, we assume spherical symmetric metric as

$$ds^2 = -e^{2f(r)}dt^2 + \frac{1}{[1-\frac{b(r)}{r}]}dr^2 + r^2 d\phi^2 \tag{1}$$

where, $r\epsilon(-\infty, +\infty)$.

To represent a wormhole , one must impose the following conditions on the metric (1) as [6] :
1) The redshift function, $f(r)$ must be finite for all values of r . This means no horizon exists in the space time .
2) The shape function, $b(r)$ must obey the following conditions at the throat $r = r_0$: $b(r_0) = r_0$ and $b'(r_0) < 1$ [these are known as Flare-out conditions].
3) $\frac{b(r)}{r} < 1$ for $r > r_0$ i.e. out of throat .
4)The space time is asymptotically flat i.e.$\frac{b(r)}{r} \rightarrow 0$ as $\mid r \mid \rightarrow \infty$.

Using the Einstein field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$, in orthonormal reference frame (with $c = G = 1$) , we obtain the following stress energy scenario,

$$8\pi\rho(r) = \frac{b'r - b}{2r^3} \tag{2}$$

$$8\pi p(r) = \frac{[1 - \frac{b}{r}]f'}{r} \tag{3}$$

$$8\pi p_{tr}(r) = (1 - \frac{b}{r})[f'' - \frac{(b'r - b)}{2r(r - b)}f' + f'^2] \tag{4}$$

where $\rho(r)$ is the energy density, $p(r)$ is the radial pressure and $p_{tr}(r)$ is the transverse pressure.

Using the conservation of stress energy tensor $T^{\mu\nu}_{;\nu} = 0$, one can obtain the following equation

$$p' + f'\rho + (f' + \frac{1}{r})p - \frac{p_{tr}}{r} = 0 \tag{5}$$

From now on , we assume that our source is characterized by the Phantom Energy with equation of state that contains a radial pressure

$$p = -k\rho \tag{6}$$

we suppose also that pressures are highly anisotropic and

$$p_{tr} = a\rho \tag{7}$$

From (5) by using (6) and (7), one can obtain

$$\rho(r)e^{(1-\frac{1}{k})f} = \frac{\rho_0}{r^{(1+\frac{a}{k})}} \tag{8}$$

where ρ_0 is an integration constant.

Taking into account equations (2)-(8), we have the following equation containing 'b' as

$$(b'r - b)^2 + (r - b)[A(b''r^2 - 3b'r + 3b) + 2B(b'r - b)] = 0 \tag{9}$$

where $\frac{1}{1-k} = A$ and $B = \frac{k+a}{k(1-k)}$.

Now to investigate whether there exists physically meaningful solutions consistent with the boundary requirements [conditions (1) to (4)], we take a general functional form of $b(r)$. We can generally express it in the form

$$b(r) = \Sigma_{n=1}^{\infty} b_n r^n + \Sigma_{m=0}^{\infty} a_m r^{-m} \tag{10}$$

since $\frac{b(r)}{r} \to 0$ as $r \to \infty$, equation (12) is consistent only when all the b_n's in $b(r)$ vanish i.e.

$$b(r) = \Sigma_{m=0}^{\infty} \frac{a_m}{r^m} \tag{11}$$

Plugging this in equation (9) and matching the coefficients of equal powers of r from both sides , we get ,

$3A = 2B$ and $48A - 12B = 36$ and these imply $k = \frac{1}{6}$, $a = \frac{1}{12}$.

Finally, one gets, the following form of b as

$$b(r) = a_0 - \frac{5a_0^2}{12r} + \frac{5a_0^3}{108r^2} + \frac{5a_0^4}{2976r^3} + \frac{587a_0^5}{964224r^4} + ...$$

Thus one gets, one parameter family of solutions.

Now the expressions for ρ and f can be obtained as

$$\rho = \frac{1}{16\pi}[\frac{5a_0^2}{6r^4} - \frac{a_0}{r^3} - \frac{15a_0^3}{108r^5} - \frac{20a_0^4}{2976r^6} - ....] \tag{12}$$

$$e^{2f} = [\frac{(\frac{5a_0^2}{6r^{\frac{5}{2}}} - \frac{a_0}{r^{\frac{3}{2}}} - \frac{15a_0^3}{108r^{\frac{7}{2}}} - \frac{20a_0^4}{2976r^{\frac{9}{2}}} - ....)}{16\pi\rho_0}]^{\frac{2k}{1-k}} \tag{13}$$

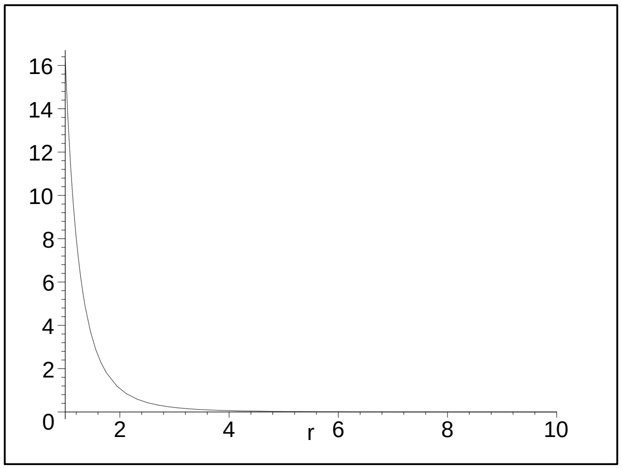

Figure 1: Energy density with respect to radial coordinate 'r' (choosing suitably the parameter)

The throat of the wormhole occurs at $r = r_0$ where r_0 satisfies the equation

$$b(r) = r \tag{14}$$

Retaining a few terms, the graph of the function $F(r) = b(r) - r$ indicates the point r_0 where $F(r)$ cuts the 'r' axis. From the graph, one can also note that when $r > r_0$, $F(r) < 0$ i.e. $b(r) - r < 0$. This implies $\frac{b(r)}{r} < 1$ when $r > r_0$.

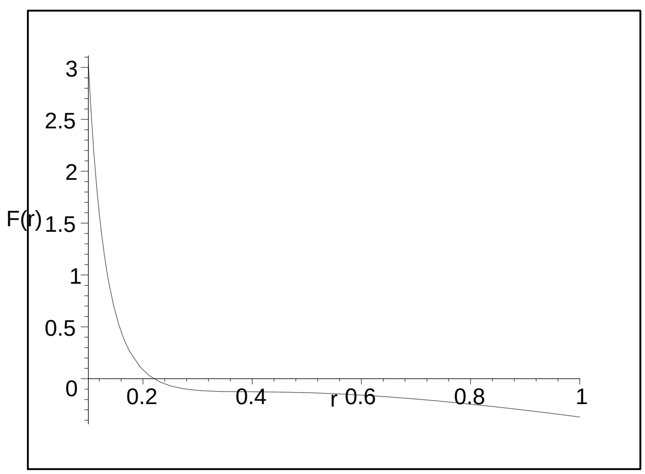

Figure 2: Throat occurs where $F(r)$ cuts 'r' axis

One can note that the redshift function $f(r)$ always finite for $0 < r_0 \leq r < \infty$ i.e. no horizon exists in the space time. Thus our solution describing a static spherically symmetric wormhole supported by the phantom energy.

According to Morris and Throne [1] , the 'r' co-ordinate is ill-behaved near the throat, but proper radial distance

$$l(r) = \pm \int_{r_0^+}^{r} \frac{dr}{\sqrt{1 - \frac{b(r)}{r}}} \qquad (15)$$

must be well behaved everywhere i.e. we must require that $l(r)$is finite throughout the space-time .

For our Model,

$$l(r) = \pm \int_{r_0^+}^{r} \frac{dr}{\sqrt{1 - \frac{1}{r}[a_0 - \frac{5a_0^2}{12r} + \frac{5a_0^3}{108r^2} + ...]}} \qquad (16)$$

Though we can not find the explicit form of the integral but one can see that the above integral is a convergent integral i.e. proper length should be finite .

The axially symmetric embedded surface $z = z(r)$ shaping the Wormhole's spatial geometry is a solution of

$$\frac{dz}{dr} = \pm \frac{1}{\sqrt{\frac{r}{b(r)} - 1}} \qquad (17)$$

One can note from the definition of Wormhole that at $r = r_0$ (the wormhole throat) Eq.(17) is divergent i.e. embedded surface is vertical there.

The embeded surface (solution of eq.(17)) in this case

$$z = \pm\sqrt{a_0}[2\sqrt{r} - \frac{7}{12}a_0 r^{-\frac{1}{2}} - \frac{199}{648}a_0^2 r^{-\frac{3}{2}} - .....] \qquad (18)$$

One can see that embedding diagram of this wormhole (retaining a few terms) in Fig-3. The surface of revolution of the curve about the vertical z axis makes the diagram complete (see fig.4).

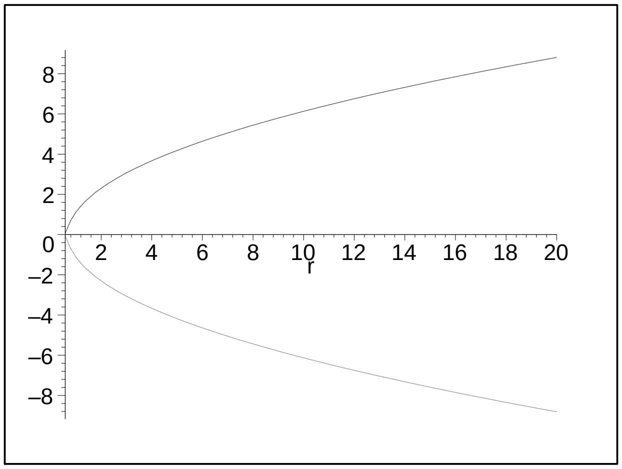

Figure 3: The embedding diagram of the wormhole (choosing suitably the parameter)

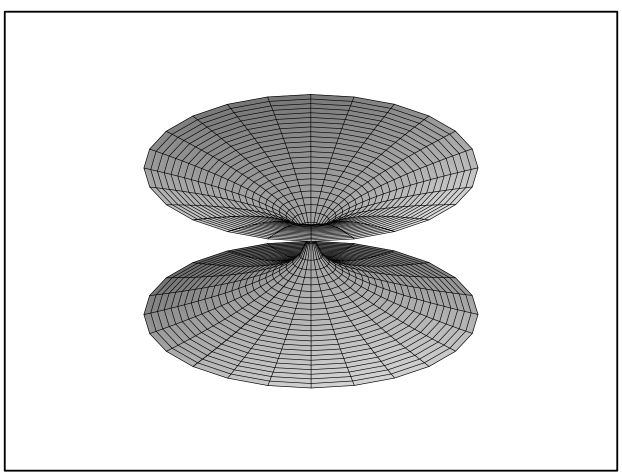

Figure 4: The full visualization of the surface generated by the rotation of the embedded curve (retaining a few terms) about the vertical z axis

In conclusion, our aim in this article has been provide a mathematical prescription for obtaining wormhole in 3D spacetime. The source is realized by phantom energy. One can also note that the equation of state is linear but highly anisotropic. We see that shape function of our model satisfies all conditions that are required to represent a wormhole. The resulting line element represents an one parameter family of geometries which contains wormholes. We note that as $r \to \infty$, the redshift function does not exist. Thus our 3D wormhole characterized by phantom energy can not be arbitrarily large. Also it may be assumed a 'cutoff' of the stress energy tensor at a junction radius 'a', where the interior wormhole metric will match to the exterior vacuum solution . We end the article with the final remarks as if an advanced engineer would able to collect sufficient amount of phantom energy, then one can imagine that they should construct wormhole with the help of several toy models [7] including this.

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